QG Omega Equation

1. Traditional form

A diagnostic expression for vertical motion is derived by manipulating the QG vorticity and thermodynamic equations. The resulting equation for frictionless, adiabatic flow is known as the QG omega equation [eq. 5.6.11 in Bluestein (1992), p. 329]

$$\left(\nabla_p^2 + \frac{f_0}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}(\zeta_g + f)\right] - \frac{R_d}{\sigma p}\nabla_p^2\left(-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}T\right)$$

where ω is the vertical velocity, $\boldsymbol{\nabla_p}$ is the gradient operator on a pressure surface, ∇_p^2 is the Laplacian operator on a pressure surface, $\boldsymbol{V_g}$ is the geostrophic wind, ζ_g is the geostrophic relative vorticity, R_d is the gas constant for dry air, p is the pressure, f is the Coriolis parameter, f_0 is the Coriolis constant (10^{-4} s^{-1}), σ is the static stability parameter (assumed constant at $2.0 \times 10^{-6}\ m^2Pa^{-2}s^{-2}$), and T is the temperature.

The right-hand-side (RHS) is calculated to determine the forcing for vertical motion at the 700 hPa level. Positive values indicate forcing for ascent. Negative values indicate forcing for descent. Note that this diagnostic calculation results in the forcing for vertical motion not the actual vertical motion because we do not solve the Laplacian on the left-hand-side.

The first term (A) on the RHS is differential advection of geostrophic absolute vorticity by the geostrophic wind. This term is evaluated using vorticity advection at 900 and 500 hPa. Finite differencing is used to evaluate the vertical derivative of vorticity advection. Cyclonic vorticity advection increasing with height and maximized at 500 hPa (term A > 0) is associated with forcing for ascent at 700 hPa. Anticyclonic vorticity advection increasing with height and maximized at 500 hPa (term A < 0) is associated with forcing for descent at 700 hPa. For the example of a 500 hPa trough located upstream of the attendant surface cyclone, term A is > 0 downstream of the 500 hPa trough and over the surface cyclone resulting in forcing for ascent over the surface cyclone. Using the QG vorticity equation $\frac{d(\zeta_g+f)}{dt} = f_0\frac{\partial\omega}{\partial p}$, we can see that 700 hPa ascent can be associated with increases of absolute vorticity at low-levels over the surface cyclone (i.e., intensification) through stretching of planetary vorticity.

The second term (B) on the RHS is the Laplacian of advection of temperature by the geostrophic wind. This term is evaluated at 700 hPa. A region of maximum warm air advection (term B > 0) is associated with forcing for ascent at 700 hPa. A region of maximum cold air advection (term B < 0) is associated with forcing for descent at 700 hPa.

The forcing for ascent associated with warm air advection and differential cyclonic vorticity advection often do not coincide. This results in a large degree of cancellation between the two terms on the RHS. Additionally, the forcing terms are not Galilean invariant. That is, if you add a uniform zonal velocity to the background flow, the magnitudes of the individual forcing terms on the RHS will change, although in reality the sum of the two forcing terms is Galilean invariant. We will now consider two forms of the QG omega equation that helps mitigate these two so-called "problems" with the traditional form.

2. Trenberth formulation

Trenberth (1978) manipulated the traditional form of the QG omega equation for frictionless, adiabatic flow resulting in the following form [eq. 5.7.40 in Bluestein (1992), p. 349]

$$\left(\nabla_p^2 + \frac{f_0}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\left[2\left(\frac{\partial \boldsymbol{V_g}}{\partial p}\cdot \boldsymbol{\nabla_p}\zeta_g\right) + \frac{\partial \boldsymbol{V_g}}{\partial p}\cdot \boldsymbol{\nabla_p} f - 2D^2\frac{\partial \theta_D}{\partial p}\right]$$

where D is the magnitude of the deformation, θ_D is the angle between the axis of dilatation (or contraction) and the x-axis, and all other symbols are the same as defined earlier.

As with the traditional form, the RHS is calculated to determine the forcing for vertical motion at the 700 hPa level. Positive values indicate forcing for ascent. Negative values indicate forcing for descent. Note that this diagnostic calculation results in the forcing for vertical motion not the actual vertical motion.

The first term (A) on the RHS is advection of geostrophic relative vorticity at 700 hPa by the thermal wind in the 900-500 hPa layer. Regions of cyclonic vorticity advection by the thermal wind (term A > 0) are located downshear (recall that the thermal wind is the vertical shear vector by definition) of cyclonic vorticity features at 700 hPa and are associated with forcing for ascent at 700 hPa. Note that term A will also be > 0 upshear of anticyclonic vorticity features. Regions of anticyclonic vorticity advection by the thermal wind (term A < 0) are located downshear of anticyclonic vorticity features at 700 hPa and are associated with forcing for descent at 700 hPa. Note that term A will also be < 0 upshear of cyclonic vorticity features. Term A tends to be the dominant forcing in the mid- and upper-troposphere, so one can get a worthy estimate of forcing for ascent from term A. This term links back to the work by R. C. Sutcliffe that highlights the importance of advection of geostrophic vorticity by the thermal wind for surface development.

The second term (B) on the RHS is advection of planetary vorticity by the thermal wind in the 900-500 hPa layer. The term is smaller and typically neglected compared to term A.

The third term (C) on the RHS represents the effects of deformation and thermal deformation. Thermal deformation is the change in deformation with pressure (or height), much like how thermal vorticity is the change in vorticity with pressure (or height). This term, while typically neglected in the mid- to upper-troposphere, can be significant near frontal zones. Term C can be expanded to $\frac{f_0}{\sigma}\left(D_2\frac{\partial D_1}{\partial p} - D_1\frac{\partial D_2}{\partial p}\right)$ to further emphasize how the forcing for vertical motion is linked to both deformation and the change of deformation with pressure (i.e., thermal deformation). Note that the components of deformation are defined as $D_1 = \frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y}$ (stretching deformation) and $D_2 = \frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}$ (shearing deformation).

The Trenberth formulation (term A in particular) is preferred over the traditional form for forecasting because the terms are Galilean invariant, do not have the cancellation "problem" between terms, and are easy to estimate by simply plotting 700 hPa geopotential height overlaid by 900-500 hPa thickness.

3. Q-vector formulation

Hoskins et al. (1978) showed that an alternative form of the QG omega equation can be derived by manipulating the QG momentum equations and QG thermodynamic equation. The resulting equation, known as the **Q**-vector form of the QG omega equation goes as [eq. 5.7.54 in Bluestein (1992), p. 352]

$$\left(\nabla_p^2 + \frac{f_0}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -2\boldsymbol{\nabla_p}\cdot\boldsymbol{Q} - \frac{R_d}{\sigma p}\beta\frac{\partial T}{\partial x}$$

where β is defined as the meridional gradient of the Coriolis parameter,

$$\boldsymbol{Q} = -\frac{R_d}{\sigma p}\begin{pmatrix}\dfrac{\partial \boldsymbol{V_g}}{\partial x}\cdot\boldsymbol{\nabla_p}T \\ \dfrac{\partial \boldsymbol{V_g}}{\partial y}\cdot\boldsymbol{\nabla_p}T\end{pmatrix} = \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$$

and all other symbols are the same as defined earlier.

As with the traditional and Trenberth forms, the RHS is calculated to determine the forcing for vertical motion at the 700 hPa level. Positive values indicate forcing for ascent. Negative values indicate forcing for descent. Note that this diagnostic calculation results in the forcing for vertical motion not the actual vertical motion.

The first term (A) on the RHS is the divergence of the **Q**-vector. Regions of **Q**-vector convergence at 700 hPa are associated with forcing for ascent at 700 hPa. Regions of **Q**-vector divergence at 700 hPa are associated with forcing for descent at 700 hPa. For tropospheric-deep synoptic systems, the **Q**-vector points in the direction of the lower tropospheric horizontal branch of the ageostrophic secondary circulation. Near baroclinic zones, **Q**-vectors pointing toward the warmer air is indicative of frontogenesis. **Q**-vectors pointing toward colder air is indicative of frontolysis.

The second term (B) on the RHS is the beta term, and is neglected because it is small relative to term A.

The **Q**-vector form of the QG omega equation has the following advantages over the traditional form when forecasting [c.f. Bluestein (1992), p. 353]:

a. Forcing function is evaluated on a single pressure level. Traditional form requires information at 3 pressure levels.
b. Forcing function is Galilean invariant.
c. No cancellation problem between forcing terms.
d. No term has been neglected.
e. **Q**-vectors can be plotted on geopotential height and temperature maps to obtain representation of forcing for vertical motion, ageostrophic circulation, and frontogenesis.

Keyser et al. (1988) introduced the vector frontogenesis (**F**-vectors), where the components of **F** in natural coordinates represent the across- (F_n) and along-isentrope (F_s)

directions (their Eq. 1.7). The component F_n is the Lagrangian rate of change of $|\nabla\theta|$ (increase in $|\nabla\theta|$ when F_n points in the $-\hat{\boldsymbol{n}}$ direction toward warmer air) and F_s is the Lagrangian rate of change of the direction of $\nabla\theta$ (counterclockwise rotation of $\nabla\theta$ when F_s points in the $\hat{\boldsymbol{s}}$ direction). Keyser et al. (1988) pointed out that the **Q**-vector is the QG analog to the **F**-vector, where the Lagrangian derivative is defined by the geostrophic flow. They suggested that it would be valuable for synoptic-dynamic diagnosis of vertical motion in weather systems to partition **Q** into its across- (Q_n) and along-isentrope (Q_s) components much like what was done with **F** in their analysis. Keyser et al. (1992) [and other subsequent studies such as Martin (1999a,b)] have demonstrated how Q_n highlights banded vertical motion about baroclinic zones and Q_s highlights cellular vertical motion associated with synoptic-scale waves. A schematic of the **Q**-vector partitioning into Q_n and Q_s in natural coordinates can be found in Martin (2006, his Fig. 6.15).

As with the full **Q**-vector form of the QG omega equation, the RHS is calculated to determine the forcing for vertical motion at the 700 hPa level associated with the Q_n and Q_s components of the full **Q**-vector. This is done by substituting Q_n (and Q_s) for **Q** in first term (A) on the RHS of the omega equation. As before, positive values indicate forcing for ascent. Negative values indicate forcing for descent. Note that this diagnostic calculation and intercomparison of the different components of **Q** results in the forcing for vertical motion not the actual vertical motion. The natural coordinate form of Q_n and Q_s can be rewritten for cartesian coordinates as follows [eq. 6.49 and 6.50 in Martin (2006), p. 179]

$$\boldsymbol{Q_n} = \left(\frac{\boldsymbol{Q}\cdot\nabla\theta}{|\nabla\theta|}\right)\frac{\nabla\theta}{|\nabla\theta|}$$

$$\boldsymbol{Q_s} = \left(\frac{\boldsymbol{Q}\cdot\left(\hat{k}\times\nabla\theta\right)}{|\nabla\theta|}\right)\frac{\hat{k}\times\nabla\theta}{|\nabla\theta|}$$

The cartesian coordinate form of Q_n and Q_s is utilized in the calculations of the QG omega equation herein. The reader is encouraged to consult Keyser et al. (1988), Keyser et al. (1992), Martin (1999a,b), and Martin (2006; section 6.4.3) for a more detailed discussion of the across- (Q_n) and along-isentrope (Q_s) components of the **Q**-vector.

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